EIGENVALUE DISTRIBUTION OF UNITARY OPERATORS ASSOCIATED TO HYPERBOLIC REFLECTION GROUPS

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ABSTRACT. We consider singular continuous measures on the unit circle associated to groups of reflections in the Poincaré unit disk. To each such measure we associate the family of orthogonal polynomials on the unit circle, the corresponding difference equation and the CMV operator, which is the unitary analog of the Jacobi operator. We use numerical methods to describe the transition in the spectral properties of the CMV operator when the Hausdorff dimension of the support of the singular measure changes continuously from 0 to 1.

1. INTRODUCTION

Recent developments in the theory of orthogonal polynomials on the unit circle have emphasized the importance of a class of unitary matrices called CMV matrices. These matrices can be regarded as the unitary analog of Jacobi matrices, hence one expects many parallels between the spectral theory of CMV matrices and of Jacobi matrices/discrete Schrödinger operators. In this paper we analyze the spectral properties of the CMV matrices associated to the probability measures on the unit circle corresponding to groups of reflections in the Poincaré unit disk.

The one-dimensional discrete Schrödinger operator is a special type of Jacobi matrix; thus, the spectral theory for discrete one-dimensional Schrödinger operators is intimately connected to the theory of orthogonal polynomials on the real line. Following this analogy, we can consider probability measures on the unit circle, the associated orthogonal polynomials on the unit circle and the corresponding unitary operators. These operators, called CMV, (as in [16] and [17]), are the unitary analogues of the Jacobi operators. Of particular interest in mathematical physics is the analysis of random CMV matrices, their spectral properties, and the distribution of their eigenvalues.

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The analysis of the eigenvalue statistics for various Hamiltonians is a relevant method to characterize the transport properties of those operators. The transition from the extended to the localized states is reflected in different level statistics: repulsion between the eigenvalues in the extended states regime and non-correlation (Poisson statistics) in the localized regime.

A transition in the distribution of the eigenvalues of the random CMV matrices with decaying random potentials was obtained in [5]. As conjectured for random Schrödinger operators, the random CMV matrices exhibit a transition in the microscopic eigenvalue statistics from no correlation (Poisson) to strong repulsion ("clock"), while at the transitional point we obtained the circular beta ensemble of the random matrix theory [12]. The Poisson eigenvalue statistics is obtained for random CMV matrices $\mathcal{C}(\{\alpha_n\}_{n\geq 0})$ with slowly decaying coefficients. The Clock statistics is obtained for CMV matrices with rapidly decaying coefficients, whereas the circular beta ensemble is obtained at the critical decay.

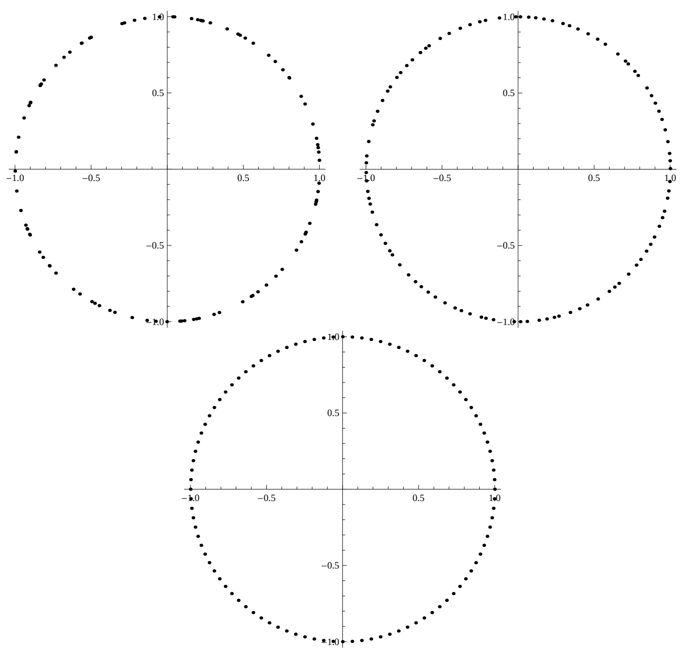

FIGURE 1. Transition in the eigenvalue statistics for CMV matrices with random decaying coefficients: from Poisson (left) to clock (right), via the CβE (middle). We plot 100 points in each case.

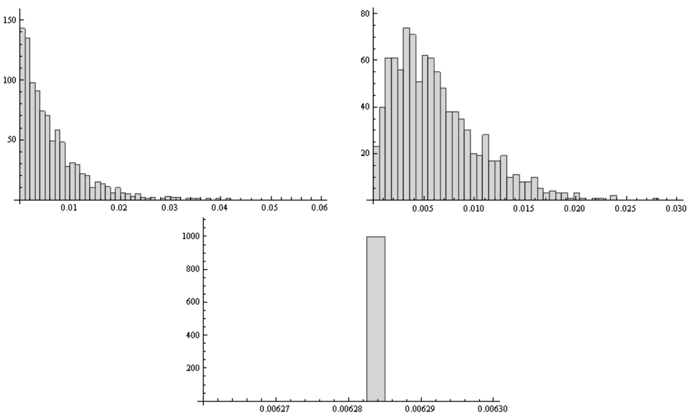

FIGURE 2. Transition in the level spacing statistics for CMV matrices with random decaying potentials: from Poisson (left) to clock (right), via the CβE (middle). We plot the histogram associated to 1000 eigenvalues in each case.

A similar transition, motivated by [6] and [7], can be observed numerically for the one-dimensional Schrödinger operator with random decaying potentials and study their eigenvalue statistics. More precisely, let $\{v_n = v_n(\omega)\}_{n\in\mathbb{Z}}$ be a sequence of independent random variables and $H = H(\omega)$ the discrete one-dimensional Schrödinger operator on $\ell^2(\mathbb{Z})$ with potential $V = \{v_n\}_n$:

$$\big(H(\omega)u\big)(n) = u(n+1) + u(n-1) + v_n(\omega)u(n)$$

We consider decaying random potential $\{v_n = v_n(\omega)\}_{n\in\mathbb{Z}}$ with expected value $\mathbb{E}\left\{|v_n(\omega)|^2\right\} \approx \frac{c}{n^\sigma}$. Motivated by the main theorem of [5], we expect that:

1. for $\sigma > 1$ (fast decay), the microscopic eigenvalue statistics is Clock;
2. for $\sigma = 1$ (critical decay), the microscopic eigenvalue statistics is given by a beta ensemble;
3. for $\sigma < 1$ (slow decay), the microscopic eigenvalue statistics is Poisson.

The plots in Figure 3 confirm this conjecture.

The transition Poisson-Clock at the level of microscopic eigenvalue statistics corresponds, at the level of the bulk spectral properties, to a transition from a pure point spectral measure to an absolutely continuous spectral measure. This phenomenon agrees with recent results on operators with clock eigenvalue statistics ([1], [8], [9]).

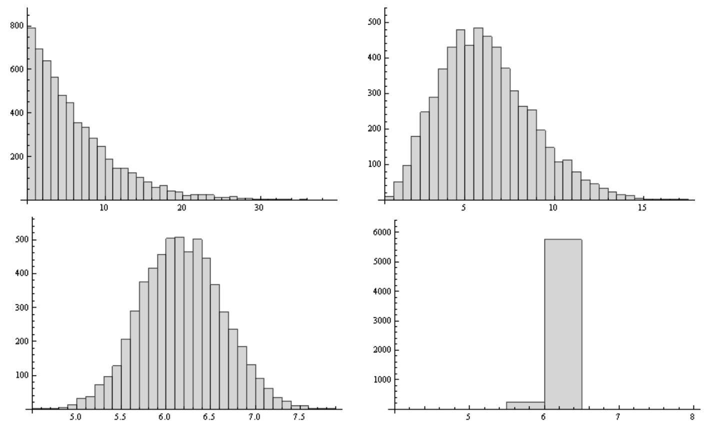

FIGURE 3. Transition in the level spacing statistics for one-dimensional discrete Schrödinger operators with random decaying potentials: from Poisson to Clock. In each case we plot the histogram associated to 6000 eigenvalues.

In this paper we will describe a similar transition for the eigenvalues of the CMV matrices associated to the limiting measures of various hyperbolic reflection groups. In Section 2 we define a class of singular continuous measures μ_θ supported on the limit set of a family of hyperbolic reflection groups. In Section 3, where we describe an algorithm for computing the Hausdorff dimension of the support of μ_θ and its moments. We conclude with Section 4, where we present a numerical analysis of the spectrum and the eigenvalue distribution of the CMV matrices $\mathcal{C}_\theta$ associated to the measures μ_θ.

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2. A Measure on the Limit Set of Hyperbolic Reflection Groups

Our main model of the hyperbolic plane is the Poincaré unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ with its hyperbolic distance function d, given explicitely, e.g., in [2, Section 7.2]. Geodesics are Euclidean circular arcs, meeting the boundary $\partial\mathbb{D}$ perpendicularly. For a given geodesic $C \subset \mathbb{D}$, the hyperbolic reflection in g is denoted by $\rho_g : \mathbb{D} \to \mathbb{D}$. This map coincides with the restriction (to $\mathbb{D}$) of the Euclidean inversion in the circle containing the geodesic g. Therefore, if $e^{i\alpha}, e^{i\beta} \in \partial\mathbb{D}$ are the end-points of g and $\alpha < \beta < \alpha + \pi$, then

$$\rho_g(z) = \frac{r^2}{\bar{z} - \bar{z}_0} + z_0 \tag{1}$$

with $z_0 = \frac{1}{\cos\gamma} e^{i\frac{\alpha+\beta}{2}}$, $r = \tan\gamma$ and $\gamma = \frac{\beta-\alpha}{2}$.

We associate to finitely many pairwise disjoint non-nested geodesics $g_1, g_2, \ldots, g_k \subset \mathbb{D}$ the discrete group $\Gamma = \Gamma_{g_1,\ldots,g_k}$ generated by the reflections r_{g_i}. The geodesics are called non-nested if, after a re-enumeration, the end-points $e^{\alpha_{2i-1}}, e^{\alpha_{2i}}$ of g_i satisfy $\alpha_1 < \alpha_2 \leq \alpha_3 < \alpha_4 \leq \cdots < \alpha_{2k} \leq \alpha_1 + 2\pi$. The closed circular arcs $C_i = C_i^{(0)} = \{e^{i\alpha} \mid \alpha_{2i-1} \leq \alpha \leq \alpha_{2i}\}$ are called *cells of generation* 0 of the reflection group Γ.

Γ acts on $\mathbb{D}$ by hyperbolic isometries. This action extends to the boundary $\partial\mathbb{D}$. The limit set of Γ is given by $\Lambda(\Gamma) = \overline{\Gamma \cdot p} \cap \partial\mathbb{D}$ (which is independent of $p \in \mathbb{D}$). $\Lambda(\Gamma)$ is a Γ-invariant subset of $\partial\mathbb{D}$. Γ is called non-elementary, if $\Lambda(\Gamma)$ contains more than two points, which is the case if $k \geq 3$. Henceforth, we assume that $k \geq 3$. If the end points of the geodesics g_l are pairwise disjoint, the limit set $\Lambda(\Gamma)$ has a Cantor-like structure. This can be seen by the following recursive definition of *cells of generation* n: Let $C_1^{(n)}, \ldots, C_N^{(n)}$ denote the cells of generation n. Then the cells of generation $n+1$ are obtained as the reflections of all cells $C_j^{(n)}$, which lie outside the cell $C_i^{(0)}$, via ρ_{g_l}, for $l = 1, \ldots, k$. Obviously, every cell of generation n contains k disjoint cells of generation $n+1$, and $\Lambda(\Gamma)$ coincides with $\bigcap_{n\in\mathbb{N}} C^{(n)}$, where $C^{(n)}$ is the union of all cells of generation n.

A Γ-invariant conformal density of dimension $\delta \geq 0$ is a finite positive measure $\mu = \mu_\Gamma$ on $\Lambda(\Gamma)$ satisfying, for all $\gamma \in \Gamma$,

$$\mu(\gamma(A)) = \int_A |\gamma'(z)|^\delta d\mu, \tag{2}$$

where γ' denotes the ordinary derivative of the map $\gamma : \partial\mathbb{D} \to \partial\mathbb{D}$ with respect to the spherical metric (i.e., angle metric in radians). This means that μ behaves like a δ-dimensional Hausdorff measures.

Many properties of conformal densities are derived from the following explicit construction, due to Patterson [14] and Sullivan [19, 20]. Let $\Gamma_0 \subset \Gamma$ be the index two subgroup of orientation preserving isometries (compositions of an even number of reflections) and $g_s(x,y) = \sum_{\gamma\in\Gamma_0} e^{-sd(x,\gamma y)}$. Then there is a critical exponent $\delta = \delta(\Gamma)$, independent of $x, y \in \mathbb{D}$, such that $g_s(x,y)$ is convergent for $s > \delta$ and divergent for $s < \delta$. Let

$$\mu_s = \frac{1}{g_s(y,y)} \sum_{\gamma\in\Gamma_0} e^{-sd(x,\gamma y)} \delta_{\gamma y}, \tag{3}$$

where δ_z denotes the Dirac delta measure at z. Since Γ_0 is geometrically finite, $g_s(x,y)$ diverges also at $s = \delta$ (see [13, Theorem 9.31]). By Helly's Theorem, there are weak limits of μ_{s_j} for certain sequences $s_j \searrow \delta$ and, because of the above mentioned divergence at the critical exponent, each such weak limit is supported on the boundary $\partial\mathbb{D}$. For the following further results we refer to [13, Chapter 9] or to the original articles of Patterson and Sullivan:

- Every weak limit is supported on $\Lambda(\Gamma)$.
- Every weak limit at $x = 0$ satisfies the transformation rule (2) (and is therefore a Γ -invariant conformal density); if we choose $x = y = 0$, all weak limits are probability measures.
- There is only one conformal density μ of dimension δ, up to scaling (this follows from the ergodicity of Γ on $\partial\mathbb{D}$ relative to the measure class of any conformal density).
- The conformal density μ has no point masses.
- The critical exponent coincides with the Hausdorff dimension of $\Lambda(\Gamma)$.

Next, we consider the special example "Symmetric pairs of pants", introduced by McMullen [11]: For $0 < \theta \leq \frac{2\pi}{3}$, let g_1, g_2, g_3 be three geodesics, symmetrically placed with respect to the origin of $\mathbb{D}$, such that the corresponding three cells C_1, C_2, C_3 of generation 0 are arcs of angle θ (the cell C_2 is bisected by the point $1 \in \partial\mathbb{D}$). Let Γ_θ denote the corresponding reflection group and Λ_θ the corresponding limit set. See [11] for the graph of the Hausdorff dimension $\theta \mapsto \delta(\Gamma_\theta) = H.dim(\Lambda_\theta)$. Let μ_θ be the unique normalized conformal density of Γ_θ.

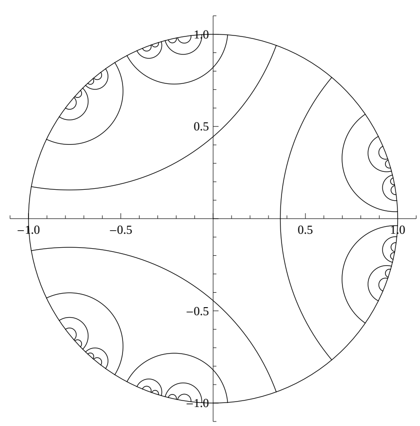

FIGURE 4. The reflection group generated by McMullen's three circle family for $\theta = \frac{5\pi}{9}$.

The family $\{\mu_\theta\}_{0<\theta<\frac{2\pi}{3}}$ represents a continuous transition (via singular continuous measures) from the purely atomic measure $\mu_0 := \frac{1}{3}(\delta_{z_1} + \delta_{z_2} + \delta_{z_3})$ with $z_k = e^{(k-2)\frac{2\pi}{3}i}$ to the normalized Lebesgue measure $\mu_{\frac{2\pi}{3}} = \lambda$:

Proposition 2.1. *Let $\theta_n, \theta \in [0, \frac{2\pi}{3}]$ and $\theta_n \to \theta$. Then the conformal densities μ_{θ_n} converge weakly to μ_θ.*

Proof. The weak convergence in the special case $\theta = 0$ follows from the symmetry of the configuration and the fact that the cells C_1, C_2, C_3 shrink to the points z_1, z_2, z_3, as $\theta_n \to 0$. The proof of [11, Theorem 3.1] implies besides continuity of the Hausdorff dimension also the weak convergence of the normalized conformal densities (see [10, Theorem 1.4]) in the case $\theta \neq 0$. If $\theta = \frac{2\pi}{3}$, the subgroup Γ_0 is a finitely generated Fuchsian group of the first kind. As noted at the end of [14], the corresponding conformal density $\mu_{\frac{2\pi}{3}}$ agrees with the normalized Lebesgue measure λ. □

3. Algorithm for Computing Hausdorff Dimension, Weights of Cells and Moments

In this section we describe McMullen's eigenvalue algorithm from [11], which provides a numerical method for approximating the Hausdorff dimension of the limiting set Λ_θ associated to the reflection group Γ_θ corresponding to the "Symmetric pairs of pants" for the angle $\theta \in (0, \frac{2\pi}{3})$ described in the previous section. We improve the algorithm in order to obtain better approximations (error $\approx 10^{-30}$) for the Hausdorff dimension. This precision is necessary in order to perform the numerical spectral analysis of the CMV matrices associated to the measures μ_θ. We also explain how this algorithm provides a method for approximating the measure μ_θ and its moments.

Let us fix an angle $\theta \in (0, \frac{2\pi}{3})$. As explained in the previous section, we start with generation 0: three geodesics g_1, g_2, and g_3 placed symmetrically around the circle. Their corresponding cells C_1, C_2, and C_3 are arcs of angle θ centered at $e^{-\frac{2\pi}{3}i}$, 1, and $e^{\frac{2\pi}{3}i}$. We denote by ρ_i the reflection with respect to g_i ($i = 1, 2, 3$). Generation 1 consists of 6 geodesics g_{12}, g_{13}, g_{21}, g_{23}, g_{31}, and g_{32} such that $\rho_i(g_{ij}) = g_j$ for any $1 \leq i, j \leq 3$ such that $i \neq j$. We relabel the geodesics from this generation g_1, g_2, g_3, g_4, g_5, and g_6 and repeat the procedure. In generation n we obtain $3 \cdot 2^n$ geodesics g_1, g_2, …, $g_{3\cdot 2^n}$. The geodesics g_{ij} from generation $(n+1)$ are obtained from $\rho_i(g_{ij}) = g_j$ for $1 \leq i \leq 3$ and $1 \leq j \leq 3 \cdot 2^n$ such that $C_i \cap g_j = \varnothing$. We relabel these geodesics g_1, g_2,

..., $g_{3\cdot 2^{n+1}}$ and repeat the procedure to obtain the generation $(n+2)$ of geodesics.

Using the same procedure, we generate points in each arc corresponding to a generation-n geodesic g_i $(1 \le i \le 3 \cdot 2^n)$. We start with three points $z_i \in C_i$, $1 \le i \le 3$; the set $\{z_1, z_2, z_3\}$ is generation 0. We define 6 points in generation 1: z_{12}, z_{13}, z_{21}, z_{23}, z_{31}, and z_{32} such that $\rho_i(z_{ij}) = z_j$ for any $1 \le i, j \le 3$ such that $i \ne j$. We relabel the points from this generation z_1, z_2, z_3, z_4, z_5, and z_6 and repeat the procedure. In generation n we obtain $3 \cdot 2^n$ points z_1, z_2, ..., $z_{3\cdot 2^n}$. The points z_{ij} from generation $(n+1)$ are obtained from $\rho_i(z_{ij}) = z_j$ for $1 \le i \le 3$ and $1 \le j \le 3 \cdot 2^n$ such that $z_j \notin C_i$. We relabel these points $z_1, z_2, \ldots, z_{3\cdot 2^{n+1}}$ and repeat the procedure to obtain the generation $(n+2)$ of points.

Therefore in generation n we get $3 \cdot 2^n$ points z_{ij}. We consider the transition matrix $T \in \mathrm{Mat}_{3\cdot 2^n \times 3\cdot 2^n}(\mathbb{C})$ defined by $T_{ij} = |\rho_i'(z_{ij})|^{-1}$ for any i, j such that $z_j \notin C_i$ and $T_{ij} = 0$ otherwise. Using Section 2 in [11], we define $\alpha(n)$ to be the unique positive number such that $\lambda(T^{\alpha(n)}) = 1$, where $\lambda(T^\alpha)$ denotes the spectral radius of the matrix T^α. The matrix elements of this matrix are obtained by raising each entry of T to power α: $(T^\alpha)_{ij} = T_{ij}^\alpha$. Then

$$\lim_{n\to\infty} \alpha(n) = H.dim(\Lambda_\theta) \tag{4}$$

Moreover, the entries of the normalized Perron-Frobenius eigenvector associated to the nonnegative matrix $T^{\alpha(n)}$ (with the largest eigenvalue 1) are the weights of the corresponding $3 \cdot 2^n$ cells defined in generation n. Thus, one can defines a probability measure μ_θ^n supported on the cells associated to the geodesics from generation n; the sequence μ_θ^n converges weakly to μ_θ.

In [15] we present an efficient algorithm for approximating the Hausdorff dimension $\alpha(\theta)$ of the set Λ_θ via the approximations $\alpha(n)$. The algorithm is based on a careful arrangement of the $3 \cdot 2^n$ cells from generation n and a combination of the power method for finding the Perron-Frobenius eigenvector and the bisection method for obtaining the approximation of the Hausdorff dimension α with the desired precision. Using this method we are able to approximate $\alpha(\theta)$ with very good precision (error $\approx 10^{-30}$):

Angle θ (in degrees)	$\alpha(\theta)$ = Hausdorff dimension of Λ_θ
0	0.000000000000000000000000000000
10	0.116009447786176637470828652 62
20	0.151183682038220751691223059 22
30	0.183983061248339186941178184 31
40	0.217765810254630012177223923 93
50	0.254382898323898222568920550 21
60	0.295546479889040808264484927 25
70	0.343333997833592693999615021 09
80	0.400676127558826851666140652 21
90	0.472189127882105566163949264 96
100	0.566098050819928624455550132 91
110	0.700550631652131273750158341 83
120	1.000000000000000000000000000000

Our result for $\theta = \frac{\pi}{6}$ agrees with the one computed by Jenkinson and Pollicott [4]. Their paper gives a different, very complex algorithm for computing the Hausdorff dimensions $\alpha(\theta)$ with high precision.

Using the previous algorithm, we can also find the moments $m_\theta(n)$ of the measure μ_θ. The symmetry of the original cells cells C_1, C_2, and C_3 implies that

$$m_\theta(3k) = m_\theta(3k+2) = 0 \tag{5}$$

If $\{z_i\}$, $1 \leq i \leq 3 \cdot 2^n$ are the the points obtained in the n-th generation using the previous recursion and $(v_1, v_2, \ldots, v_{3 \cdot 2^n})^T$ is the positive Perron-Frobenius eigenvector associated $T^{\alpha(\theta)}$, the eigenvalue algorithm gives the following formula for approximating the nonzero moments:

$$m_\theta(3k+1) = \lim_{n \to \infty} 6 \sum_{i=1}^{2^{n-1}} v_i \cos[3\, k\, z_{3 \cdot 2^{n-1}+i}] \tag{6}$$

This proposition allows us to plot the moments of the measures μ_θ (see Figure 5).

We observe an interesting transition from patterns with rare oscillations and large amplitudes (for small angles θ) to patterns with very dense oscillations and small amplitudes (for large angles θ).

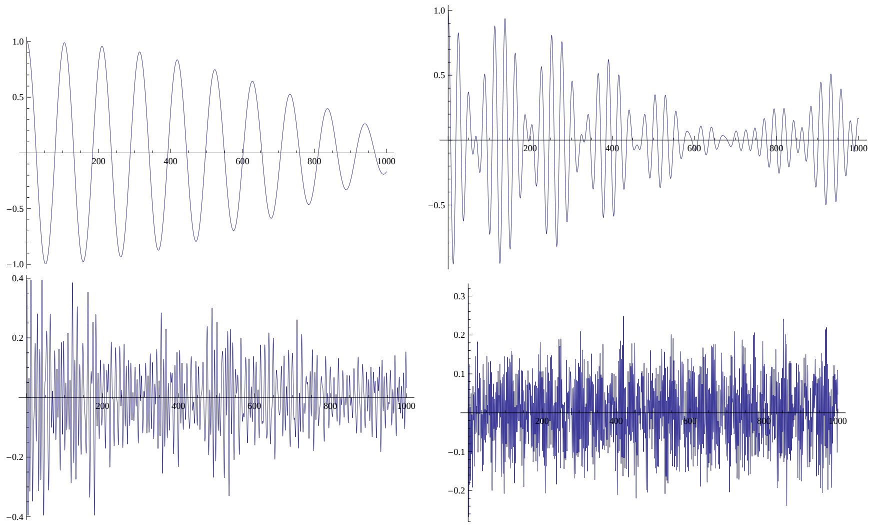

FIGURE 5. Plots of the nonzero moments $m_\theta(3k+1)$, $1 \le k \le 1000$ of the measure μ_θ for $\theta = \frac{\pi}{6}$ (first line, left), $\theta = \frac{\pi}{3}$ (first line, right), $\theta = \frac{5\pi}{9}$ (second line, left) and $\theta = \frac{119\pi}{180}$ (second line, right).

4. The CMV Operators Associated the Hyperbolic Reflection Groups and the Distribution of Their Eigenvalues

A 2002 paper of Cantero, Moral and Velázquez [3] emphasized the importance of a new class of unitary matrices called now CMV matrices. The characteristic polynomials of the finite truncations of the CMV matrices with spectral measure μ are the orthogonal polynomials on the unit circle associated to the measure μ (see for example the monographs [16], [17], and the review article [18]). This fact provides an immediate connection between the spectral theory for CMV matrices and the orthogonal polynomials on the unit circle.

The CMV matrix is a five-diagonal matrix realization for the unitary operator $z \to zf(z)$ on $L^2(\mathbb{T};\mu)$, where μ is a non-trivial probability measure on the unit circle $\mathbb{T}$ (we call a measure non-trivial if it is not supported on finitely many points). For any such measure μ we can apply the Gram-Schmidt procedure to the set of polynomials $\{1, z, z^2, \ldots\} \in L^2(\mathbb{T},\mu)$ and get the set of monic orthogonal polynomials $\{\Phi_0(z,d\mu),\, \Phi_1(z,d\mu),\, \Phi_2(z,d\mu),\, \ldots\} \in L^2(\mathbb{T};\mu)$.

These polynomials obey the recurrence relation

$$\Phi_{k+1}(z,d\mu) = z\Phi_k(z,d\mu) - \overline{\alpha}_k \Phi_k^*(z,d\mu) \qquad k \ge 0 \tag{7}$$

where, for $\Phi_k(z) = \sum_{j=0}^{k} b_j z^j$, the reversed polynomial $\Phi_k^*(z)$ is defined by $\Phi_k^*(z) = \sum_{j=0}^{k} \bar{b}_{k-j} z^j$. The recurrence coefficients $\{\alpha_n\}_{n\geq 0}$ are called Verblunsky coefficients; they are complex numbers of absolute value < 1.

If we apply the Gram-Schmidt algorithm to the sequence $\{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$ we get the set of functions $\{\chi_0(z), \chi_1(z), \chi_2(z), \ldots\}$, which is a basis of $L^2(\mathbb{T};\mu)$. The CMV matrix associated to the measure μ is the matrix representation of the operator $f(z) \to zf(z)$ on $L^2(\mathbb{T};\mu)$. It has the form:

$$\mathcal{C} = \begin{pmatrix} \bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \ldots \\ \rho_0 & -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots \\ 0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \ldots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \ldots \\ 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \tag{8}$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$

Note that the Jacobi matrices obtained in a similar way for orthogonal polynomials on the real line are tri-diagonal matrices. As in the case of orthogonal polynomials on the real line, an important connection between CMV matrices and monic orthogonal polynomials is

$$\Phi_n(z) = \det(zI - \mathcal{C}^{(n)}) \tag{9}$$

where $\mathcal{C}^{(n)}$ is the upper left $n \times n$ corner of $\mathcal{C}$.

If $|\alpha_{n-1}| = 1$, then the CMV matrix decouples between $(n-1)$ and n. The upper left corner is an $(n \times n)$ unitary matrix $\mathcal{C}^{(n)} = \mathcal{C}^{(n)}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ and we have $\mathcal{C} = \mathcal{C}^{(n)} \oplus \tilde{\mathcal{C}}$, where $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}(\alpha_n, \alpha_{n+1}, \ldots)$ is an (infinite) CMV matrix.

Using the moments $m_\theta(n)$, which can be approximated with high precision using the algorithm outlined in the previous section, we can construct the orthogonal polynomials $\Phi_n(z) = \Phi_n(z,\theta)$ associated to the measure μ_θ. The Verblunsky coefficients $\{\alpha_n = \alpha_n(\theta)\}$ are obtained from

$$\alpha_n = -\overline{\phi_{n+1}(0)} \tag{10}$$

We can then write the CMV matrix $\mathcal{C} = \mathcal{C}_\theta(\alpha_0, \alpha_1, \ldots)$. For any positive integer n we choose a complex number α_n such that $|\alpha_n| = 1$, which gives a unitary $n \times n$ matrix $\mathcal{C}^{(n)}$. We can plot the eigenvalues of $\mathcal{C}^{(n)}$ for various values of θ and get plots that show the transition towards strong repulsion of the eigenvalues (Clock distribution) as $\theta \to \frac{2\pi}{3}$.

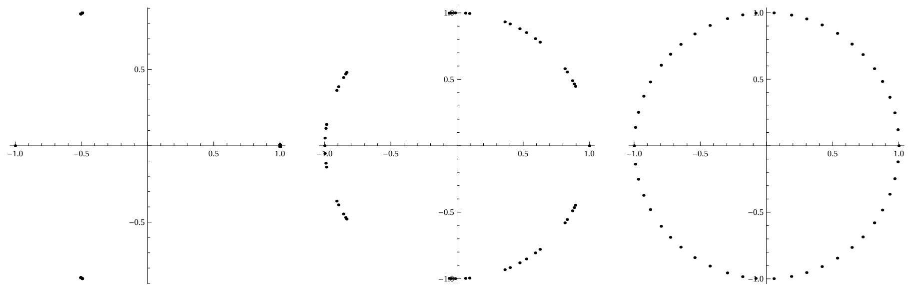

FIGURE 6. Plots of the eigenvalues of the CMV matrix $\mathcal{C}_\theta$ for $\theta = \frac{\pi}{18}$ (left), $\theta = \frac{5\pi}{9}$ (middle) and $\theta = \frac{119\pi}{180}$ (right).

The universality of the clock eigenvalue distribution was obtained for orthogonal polynomials on the real line [9] and for ergodic Jacobi matrices with absolutely continuous spectrum [1]. We presented here numerical evidence for the transition towards clock eigenvalue distribution for a class of unitary matrices with singular continuous spectral measures. We expect that this phenomenon holds in most cases where one considers operators associated families of measures that interpolate between pure point measures and absolutely continuous measures.

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